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Super elliptic curves

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Abstract

A detailed study is made of super elliptic curves, namely super Riemann surfaces of genus one considered as algebraic varieties, particularly their relation with their Picard groups. This is the simplest setting in which to study the geometric consequences of the fact that certain cohomology groups of super Riemann surfaces with odd spin structure are not freely generated modules. The divisor theory of Rosly, Schwarz, and Voronov gives a map from a supertorus to its Picard group Pic, but this map is a projection, not an isomorphism as it is for ordinary tori. The geometric realization of the addition law on Pic via intersections of the supertorus with superlines in projective space is described. The isomorphisms of Pic with the Jacobian and the divisor class group are verified. All possible isogenies, or surjective holomorphic maps between supertori, are determined and shown to induce homomorphisms of the Picard groups. Finally, the solutions to the new super Kadomtsev–Petviashvili hierarchy of Mulase–Rabin which arise from super elliptic curves via the Krichever construction are exhibited.

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1. Introduction

The theory of elliptic curves [1,2] is not only a rich and fascinating subject in its own right, but a confluence of several major branches of mathematics and a source of simple and explicitly computable examples in each. These include Riemann surfaces, algebraic groups, Abelian varieties, divisor theory, Diophantine equations, mapping class groups, and automorphic functions. The simple modular properties of the torus are of particular importance in conformal field theory, owing to the sewing axioms, by virtue of which

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modular invariance on the torus guarantees this invariance at higher genus, and in the related theory of elliptic genera.

The study of super elliptic curves, meaning super Riemann surfaces of genus one considered as algebraic varieties, was initiated in [3,4] with the use of superelliptic functions and super theta functions to embed supertori in projective superspace as the sets of zeros of explicit polynomial equations, generalizing the Weierstrass equation for an elliptic curve. Missing from this work was any discussion of the group law on a superelliptic curve. Associated to any Riemann surface is its Picard group or Jacobian, the group of line bundles of degree zero on the surface under tensor product. A torus is itself a group because it is isomorphic to its Jacobian via the classical Abel map. The situation for supertori is more complicated because the Abel map turns out to be a projection rather than an isomorphism. The proof of this fact and extensive discussion of its consequences for the theory of superelliptic curves are the subjects of this paper.

We study specifically the supertorus with odd spin structure given informally (a more precise definition follows) as the quotient $M = \mathbb{C}^{1,1}/G$ of the complex superplane with coordinates (z, θ) by the supertranslation group G generated by the transformations

$$
\begin{aligned}
&T : z \to z + 1, \quad \theta \to \theta, \\
&S : z \to z + \tau + \theta\delta, \quad \theta \to \theta + \delta.
\end{aligned} \tag{1}
$$

The odd spin structure is of interest precisely because of the presence of the odd modular parameter δ in addition to the usual even one τ (with the modulus τ and the theta functions, we will tolerate some exceptions to the standard convention that Greek letters denote odd quantities while Roman letters denote even ones). Meromorphic functions on the supertorus are just meromorphic functions $F(z, \theta)$ on $\mathbb{C}^{1,1}$ which are G-invariant, or superelliptic. In particular, the cohomology group $H^0(M, \mathcal{O})$ consisting of global holomorphic functions is easily shown to be the set of functions $a + \alpha\theta$ with constant coefficients a, α such that $\alpha\delta = 0$. For even functions, a should be even and α odd. Owing to the constraint on α, this is *not* simply the vector superspace $\mathbb{C}^{1,1}$ with basis $\{1, \theta\}$; it is indeed a module over the Grassmann algebra Λ containing all our odd parameters, but this module is not freely generated. This situation occurs generically for super Riemann surfaces with odd spin structure [5,6] and its implications are not well understood in general. The primary motivation for this work was to study them in this simplest case, in which complete, explicit calculations are possible and illuminating.

One is so accustomed to the fact that a sheaf cohomology group $H^i(M, \mathcal{F})$ typically carries the structure of a finite-dimensional vector space that one forgets that the proof is nontrivial [7]. Certainly the existence of this structure is so central to geometric applications of cohomology that one would hardly know where to begin without it: the Riemann-Roch theorem is only the simplest of the tools designed to compute the dimensions of these vector spaces. Such tools only generalize in the super case for generic *even* spin structures, the "normal case" considered in [8]. The lack of a super vector space structure causes difficulties in the theory and applications of super Riemann surfaces whenever a basis for a cohomology space of functions, differentials, or deformations would be desirable. (One should note that because Λ is a Grassmann algebra over $\mathbb{C}$, the cohomology groups do have vector space structures over $\mathbb{C}$. However, because one may wish to vary the Grassmann algebra, it is the module structure over Λ that is of interest.) Certainly the super Riemann-Roch theorem holds only in the normal case or under additional assumptions. In the application to superstrings, bases for spaces of holomorphic differentials of various weights are normally used to express the superdeterminants appearing in the path integral measure and in the expressions for amplitudes used in finiteness and unitarity proofs. These analyses are considerably more complicated when such free bases do not exist [9]. For an infinite-dimensional example, the geometry associated to the super KP hierarchies [10,11] is described in terms of a super Grassmannian of vector subspaces of, say, functions on the supercircle [12] with the Krichever map sending a supercurve M equipped with a line bundle $\mathcal{L}$ and various coordinate choices to the vector subspace given by a direct sum of cohomology groups

$$\bigoplus_{n=0}^{\infty} H^0(M, \mathcal{L} \otimes \mathcal{O}(nP)).$$

These are the global sections of $\mathcal{L}$ holomorphic except for poles of arbitrary order at a marked point P. The arguments in [10] show (although it was not stated clearly there) that, although the summands may not be freely generated for small n, they become so for n sufficiently large, so that the sum is indeed a super vector space and the Krichever map makes sense.

This paper concentrates on how the non-free character of the cohomology affects the geometry of a superelliptic curve, particularly its relation to its Jacobian. Section 2 develops the basics of function theory. We exhibit the building blocks for the explicit construction of functions, the super analogues of Weierstrass $\wp$ functions and theta functions, as well as deriving the general constraints on the divisor of a superelliptic function. Because the canonical bundle of a superelliptic curve is trivial, this analysis applies to meromorphic differentials of all weights as well as to functions. In Section 3 we explicitly compute the Picard group (group of line bundles), the Jacobian (space of linear functionals on holomorphic 1/2-differentials, modulo periods), and the divisor class group (divisors modulo divisors of functions) of a superelliptic curve, verifying that they are all isomorphic. This isomorphism has been proven for all super Riemann surfaces in the normal case [8], but not more generally thus far. The Abel map from the curve to its Jacobian is obtained and observed to be a projection π: it takes the quotient of the curve by the relation $(z,\theta) \equiv (z + \alpha\delta, \theta)$ for all α. The origin of this extra identification is traced to the necessity of abelianizing the nonabelian group $\mathbb{C}^{1,1}$ in order for the quotient to admit a group structure. Section 4 shows that, modulo this identification and an ambiguity in the choice of identity element, the group operation on the Jacobian can be performed geometrically on the curve by intersecting it with special planes in the standard superprojective embedding. Section 5 determines all the isogenies of superelliptic curves. These are surjective holomorphic mappings between supertori. For elliptic curves one proves that they are necessarily homomorphisms in the group structure. Here, since a superelliptic curve does not carry the group structure of its Jacobian, the best one can do is to show that an isogeny induces a homomorphism of the Jacobians via the projection π. We also study isogenies of a superelliptic curve to itself and show that a nonsplit curve admits only trivial endomorphisms. Section 6 contains an application of these results to the new super KP system discovered by Mulase and the author [10,11]. This system of nonlinear PDEs for the coefficients of a pseudosuperdifferential operator describes, via the Krichever construction, the deformation of a line bundle $\mathcal{L}$ over an algebraic supercurve by certain commuting flows in the Jacobian. The pseudodifferential operator is closely related to a special section of $\mathcal{L}$ called the Baker–Akhiezer function. The algebraic supercurves involved are generally not super Riemann surfaces except in the special case of genus one. In this exceptional case we can construct explicit solutions to the super KP system describing flows in the Jacobian of a superelliptic curve, in terms of Weierstrass elliptic functions. The result can be presented as an isomorphism between a ring of meromorphic functions on the superelliptic curve and a ring of supercommuting differential operators [10,14]. It generalizes the classical result that the operators

$$Q = \frac{d^2}{dx^2} - 2\wp(x+a), \tag{2}$$

$$P = Q_+^{3/2} = \frac{d^3}{dx^3} - 3\wp(x+a)\frac{d}{dx} - \frac{3}{2}\wp'(x+a) \tag{3}$$

arising from an elliptic curve generate a commutative ring. The parameter a should be viewed as a coordinate on the Jacobian and varies linearly with the flow parameters. A new feature of the super case is that the supercommutativity of the ring depends upon the fact that the theta function satisfies the heat equation. Section 7 contains conclusions and directions for further research. An Appendix briefly considers the problem of finding rational points on superelliptic curves. Here the nilpotent elements of Λ linearize the problem to locating rational points on the (co)tangent line to an elliptic curve at a rational point. Throughout this paper, computations which employ standard methods are nevertheless given in considerable detail, so as to remove any mystery from the supermodulus δ and display clearly the role it plays in modifying the classical results.

Before proceeding, let us return to the precise definition of the superelliptic curves we study. We fix a finite-dimensional complex Grassmann (exterior) algebra Λ in which δ is an odd element and τ an even one with $\operatorname{Im} \tau_{\rm rd} > 0$. (Throughout this paper the subscript "rd" on a Grassmann variable, supermanifold, supergroup, etc. denotes the reduction of this object by modding out the ideal of nilpotents in Λ or in the structure sheaf.) We adopt the standard sheaf-theoretic treatment of supermanifolds [15] within which we are really dealing with families of superelliptic curves over the parameter superspace $\mathcal{B} = (\mathrm{pt}, \Lambda)$. Our covering space, informally denoted $\mathbb{C}^{1,1}$, is really the trivial family $\mathbb{C}^{1,1} \times \mathcal{B}$, meaning the complex plane $\mathbb{C}$ equipped with the structure sheaf $\mathcal{O}_{\mathbb{C}} \otimes \Lambda[\theta]$, where $\Lambda[\theta]$ is the larger Grassmann algebra whose generators are θ and the generators of Λ. The family of superelliptic curves M over $\mathcal{B}$ is the quotient of this family by the group G, meaning the following. The reduced space of M is the standard torus $M_{\rm rd}$ with modular parameter τ_{rd}. The structure sheaf of M assigns to any open set U of M_{rd} the following ring $\mathcal{O}_U$. U is covered by a collection of connected open sets U_i of $\mathbb{C}$. To each element g in G there corresponds a transformation g_{rd} in the reduced group G_{rd} generated by

$$T_{rd} : z \to z + 1, \qquad S_{rd} : z \to z + \tau_{rd}, \tag{4}$$

which maps each U_i to some (possibly the same) U_j. For $\mathcal{O}_U$ we take all collections of functions $\{F_i(z,\theta) \in \mathcal{O}_{U_i}\}$ which are G-invariant in the sense that $F_j(z,\theta) = gF_i(z,\theta)$ whenever $U_j = g_{rd}U_i$, $g \in G$. Here g acts on functions via Taylor expansion in nilpotents as usual: $F(z+\tau+\theta\delta, \theta+\delta)$ means $F(z+\tau,\theta) + \theta\delta\partial_z F(z+\tau,\theta) + \delta\partial_\theta F(z+\tau,\theta)$. If τ has a nilpotent part then the last expression is defined by further Taylor expansion in this nilpotent part. The statement that $\rho : M \to \mathcal{B}$ is a family means that there is a pullback map of the functions Λ on $\mathcal{B}$ to functions on M; the elements of Λ play the role of global constant functions on M and as such all the cohomology groups of M are modules over Λ (or its even part if the sheaf is purely even or odd).

For those readers less comfortable with sheaf-theoretic language, which often includes the author, we can consider the set of Λ-valued points of M rather than M itself. This is the set of (even) maps $\mathcal{B} \to M \xrightarrow{\rho} \mathcal{B}$ for which the composed map $\mathcal{B} \to \mathcal{B}$ is the identity. For each point of M this is an evaluation of functions at that point by assigning even and odd values from Λ to the coordinates z and θ respectively. That is, it is just an abstract description of the Λ-supermanifolds of [8], or the supermanifolds of DeWitt [16] or Rogers [17], which are genuine sets of points with Grassmann-valued coordinates. The Picard and Jacobian groups as defined here naturally appear as such sets of Λ-valued points and will be discussed as such; our constructions can be translated into pure sheaf-theoretic terms by those readers with the sophistication to prefer this viewpoint.

The choice of Grassmann algebra will usually be left open, but two cases are worth distinguishing. One is the case in which δ is one of the generators of Λ. The most important example is the two-dimensional algebra having δ as its only generator (plus unity); if we let τ run through the upper half-plane this gives the universal Teichmüller family of supertori (apart from the identification of $\pm\delta$). The other is the general case in which δ is an element of Λ but not necessarily a generator. Such a family is a pullback of the universal family by a map of the base spaces, which indeed pulls back δ to some element of Λ, e.g. $\delta = \beta_1\beta_2\beta_3$ in terms of generators β_i. The most important distinction between these cases is that when δ is a generator it annihilates only multiples of itself, while in general it may annihilate other elements as well, e.g. multiples of β_1 in the above example.

2. Basic function theory

In order to construct explicit functions and sections of bundles on the supertorus M, in particular the Baker–Akhiezer function appearing in super KP theory, we need the building blocks corresponding to the Weierstrass elliptic function $\wp(z;\tau)$ and the theta function $\Theta(z;\tau)$ (the capital letter is used for theta functions in this paper to avoid confusion with the coordinate θ) introduced in [4].

The super Weierstrass function is

$$R(z,\theta;\tau,\delta) = \wp(z;\tau+\theta\delta) = \wp(z;\tau) + \theta\delta\dot{\wp}(z;\tau), \tag{5}$$

where by convention a dot denotes ∂_τ while a prime will mean ∂_z. It is superelliptic, as are its supercovariant derivatives $D^n R$, where $D = \partial_\theta + \theta\partial_z$ commutes with the generators of G and satisfies $D^2 = \partial_z$. These functions provide the standard embedding of M in projective superspace which we will recall in Section 4.

Similarly, our super theta function will be

$$H(z,\theta;\tau,\delta) = \Theta(z;\tau+\theta\delta). \tag{6}$$

The ordinary theta function appearing here is the one often denoted $\Theta[\,{}^{1/2}_{1/2}\,](z;\tau)$, which corresponds to the odd spin structure. It has a simple zero at $z=0$ and the other lattice points, and satisfies

$$\begin{aligned} &\Theta(z+1;\tau) = -\Theta(z;\tau) = \Theta(-z;\tau), \\ &\Theta(z+\tau;\tau) = -e^{-\pi i\tau - 2\pi i z}\Theta(z;\tau). \end{aligned} \tag{7}$$

As a result, the super theta function satisfies

$$\begin{aligned} &H(z+1,\theta) = -H(z,\theta) = H(-z,\theta), \\ &H(z+\tau+\theta\delta,\theta+\delta) = -e^{-\pi i\tau - \pi i\theta\delta - 2\pi i z}H(z,\theta), \end{aligned} \tag{8}$$

where the moduli dependence of H has been suppressed. The relation between Θ and $\wp$ is [18]

$$\frac{d^2}{dz^2}\log\Theta(z;\tau) = -\wp(z;\tau) + q, \quad q = \frac{\Theta'''(0;\tau)}{3\Theta'(0;\tau)}. \tag{9}$$

The first derivative $\partial_z \log\Theta$ is nearly elliptic, being invariant under $z \to z+1$ and changing by an additive constant under $z \to z+\tau$. Since this is also the behavior of θ according to (1), we can form the superelliptic combination [19]

$$\sigma(z,\theta;\tau,\delta) = \theta + \frac{\delta}{2\pi i}\frac{d}{dz}\log\Theta(z;\tau), \tag{10}$$

which reduces to θ in the split case where $\delta = 0$. This function will be of particular importance in view of the fact that it is holomorphic in the split case (when cohomology is freely generated) but only meromorphic otherwise.

To describe the meromorphic functions on M and construct them from the building blocks above, we turn to the study of divisor theory. In the usual Cartier divisor theory, a divisor would be a subvariety of codimension $(1,0)$, hence dimension $(0,1)$, given locally by an even equation $F(z,\theta) = 0$. The fact that such divisors are not points breaks the strong analogy between elliptic and superelliptic curves. It was the great insight of Rosly, Schwarz, and Voronov [8] (see also [20]) to make use of the covariant derivative D (the superconformal structure) which exists locally on any super Riemann surface to define divisors of codimension $(1,1)$—points—via the simultaneous solutions of

$$F(z,\theta)=0, \quad DF(z,\theta)=0. \tag{11}$$

For any even function F for which the reduced function $F_{\mathrm{rd}}(z)$ is not identically zero, a point (Λ-valued!) (z_0,θ_0) satisfying these equations is called a principal zero of F. If we write $F(z,\theta)=f(z)+\theta\phi(z)$ and assume that $(z_0)_{\mathrm{rd}}$ is a simple zero of f_{rd} (in this case we are discussing a principal simple zero of F), this amounts to the statements

$$f(z_0)=0, \quad \theta_0=-\phi(z_0)/f'(z_0). \tag{12}$$

A principal pole of F is a principal zero of $1/F$. A formal sum of points $\sum n_iP_i$ is a divisor of F provided that in a chart containing $P_i=(z_i,\theta_i)$ we can write

$$F(z,\theta)=E(z,\theta)\prod_i(z-z_i-\theta\theta_i)^{n_i}, \tag{13}$$

where the product is over the P_i contained in the chart and E is holomorphic with $E_{\mathrm{rd}}\neq 0$ in this chart [it may not be possible to separate all the points P_i because the corresponding reduced points $(z_i)_{\mathrm{rd}}$ may coincide]. A subtlety is that a single function may have more than one divisor if its zeros and poles are not simple. For example, on $\mathbb{C}^{1,1}$, $F=(z+a)^2=z(z+2a)$ with nilpotent even constant a satisfying $a^2=0$ has the two distinct divisors of zeros $2(-a,0)$ and $(0,0)+(-2a,0)$ as well as others. On the supertorus, $R(z,\theta)$ has a principal double pole at $(0,0)$ and two simple zeros. The super theta function $H(z,\theta)$, actually a section of a bundle rather than a function, has a principal simple zero at $(0,0)$.

We now derive the necessary and sufficient condition for a divisor $\sum n_iP_i$ to be a divisor of some meromorphic function F on M: the sum of the P_i with multiplicity must differ from a lattice point by $(\alpha\delta,0)$ for some constant α, namely

$$\sum_i n_i\theta_i=n\delta, \qquad \sum_i n_iz_i=m+n\tau+\alpha\delta, \tag{14}$$

for integers m,n. Of course, the total degree $\sum_i n_i$ must also vanish because it vanishes for the divisor of the reduced function on the torus M_{rd}.

The proof of the necessity follows the classical and elementary proof for elliptic curves [1] by integrating $DF/F=D\log F$ around a period parallelogram as shown in Fig. 1, chosen to avoid the points of the divisor. An easy computation shows that near a principal pole or zero where F behaves as $(z-z_i-\theta\theta_i)^{n_i}$, we have

$$\frac{DF}{F}\sim\frac{n_i(\theta-\theta_i)}{z-z_i-\theta\theta_i}=\frac{n_i(\theta-\theta_i)}{z-z_i}, \tag{15}$$

plus holomorphic terms. Then we evaluate the following two contour integrals (for details on the definition of super contour integration, see [21-23]; for closed contours

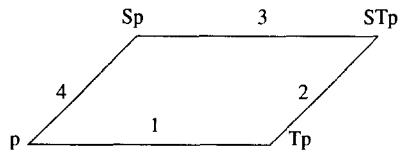

Fig. 1. The period parallelogram, an integration contour for the proof of sum rules for the divisor of a superelliptic function. Except for orientation, sides 1 and 3 are related by the supertranslation S, sides 2 and 4 by T.

it is simply Berezin integration over θ followed by ordinary contour integration; for an open contour lying in a simply connected region in which F is holomorphic, it is the change in an antiderivative Φ, with $D\Phi = F$, between the endpoints):

$$\oint \theta \frac{DF}{F} dz = \sum_i \oint \frac{-n_i \theta \theta_i}{z - z_i} dz\, d\theta = 2\pi i \sum_i n_i \theta_i, \tag{16}$$

and similarly

$$\begin{aligned}\oint z \frac{DF}{F} dz &= \sum_i \oint \frac{n_i z \theta}{z - z_i} dz\, d\theta = \sum_i n_i \oint \left(1 + \frac{z_i}{z - z_i}\right) dz \\ &= 2\pi i \sum_i n_i z_i.\end{aligned} \tag{17}$$

Next we evaluate the integrals over each side of the parallelogram and use the fact that F is the same on opposite sides by superellipticity. For the first integral we note that θ is the same on sides 2 and 4, which have opposite orientations, so those contributions cancel, while sides 1 and 3 are related by $\theta \to \theta + \delta$. The Jacobian factors relating these integrals are unity, which is also clear from the antiderivative definition and the fact that D commutes with the generators of G. Hence these contributions sum to

$$\oint \theta \frac{DF}{F} dz = \int_1 -\delta \frac{DF}{F} dz = -\delta \int_1 D \log F dz = 2\pi i n \delta, \tag{18}$$

the point being that only the reduced part of $\log F$ is multivalued, the nilpotent part involving derivatives of log via the Taylor expansion. Comparing with the previous evaluation of the integral gives the sum rule for θ_i. For the z integral things are slightly more complicated. Sides 1 and 3 are related by $z \to z + \tau + \theta\delta$, sides 2 and 4 by $z \to z + 1$. Making these substitutions gives

$$\begin{aligned}\oint z \frac{DF}{F} dz &= \int_1 (-\tau - \theta\delta) \frac{DF}{F} dz + \int_2 \frac{DF}{F} dz \\ &= 2\pi i (m + n\tau) + \delta \int_1 \theta \frac{DF}{F} dz,\end{aligned} \tag{19}$$

where the last integral can have any odd value. Calling it $-2\pi i \alpha$, we obtain the sum rule for z_i.

To show the sufficiency, we construct a function having any given divisor satisfying the sum rules in terms of the super theta function. First we note the effect of a supertranslation on the divisor of a function: if $F(z,\theta)$ has the behavior $(z-z_i-\theta\theta_i)^{n_i}$ corresponding to a principal zero or pole at (z_i,θ_i), then

$$F(z-a-\theta\epsilon,\theta-\epsilon)\sim[z-(z_i+a+\theta_i\epsilon)-\theta(\theta_i+\epsilon)]^{n_i}, \tag{20}$$

shifting the zero or pole to $(z_i+a+\theta_i\epsilon,\theta_i+\epsilon)$. The odd coordinates of the divisor are shifted uniformly by ϵ, the even coordinates uniformly by a but also nonuniformly by a term proportional to the odd coordinates. This changes the sum of the z_i by a multiple of the sum of the θ_i, which is a multiple of δ, consistent with the sum rule for z_i. In particular, the theta function $H(z-z_i-\theta\theta_i,\theta-\theta_i)$ is holomorphic with a principal simple zero at (z_i,θ_i).

Unfortunately, this theta function is not convenient for our purposes since it does not transform by a mere phase under the group G. As a consequence of the commutation relations of supertranslations, the generator S sends it to a phase times $H(z-z_i-\theta\theta_i-2\delta\theta_i,\theta-\theta_i)$. However, the function $H(z-z_i-\theta\theta_i,\theta+\theta_i)$ also has a principal simple zero at (z_i,θ_i) and transforms as

$$\begin{aligned} &SH(z-z_i-\theta\theta_i,\theta+\theta_i)\\ &\quad=-e^{-\pi i[\tau+(\theta+\theta_i)\delta+2(z-z_i-\theta\theta_i)]}H(z-z_i-\theta\theta_i,\theta+\theta_i). \end{aligned} \tag{21}$$

This remedy of changing the relative sign in $\theta-\theta_i$ amounts to the usual replacement of a SUSY generator by a SUSY covariant derivative.

Let us suppose first that $\sum_i n_iP_i$ is a degree-zero divisor for which the P_i sum exactly to a lattice point, with no remainder $\alpha\delta$. By adding the fictitious points $(0,0)-(m+n\tau,n\delta)$ we can assume that the P_i sum to zero without changing the divisor on M. Then a superelliptic function with this divisor is

$$F(z,\theta)=\prod_i[H(z-z_i-\theta\theta_i,\theta+\theta_i)]^{n_i}. \tag{22}$$

Its invariance under the generators of the group G is easily checked using the relation (21) and the sum rules (14).

The simplest example of a degree-zero divisor satisfying the sum rules with a nontrivial remainder $\alpha\delta$ is $\Delta=(\alpha\delta,0)-(0,0)$. A meromorphic function with this divisor is easily constructed from the function σ introduced in Eq. (10), namely

$$F_\Delta(z,\theta)=1-2\pi i\alpha\sigma(z,\theta)=(1-2\pi i\alpha\theta)\left(1-\alpha\delta\frac{d}{dz}\log\Theta(z;\tau)\right), \tag{23}$$

where the second form shows the behavior $1-\alpha\delta/z$ near $z=0$ dictated by the divisor. Now, given an arbitrary divisor satisfying the sum rules, subtracting the divisor Δ produces one which sums exactly to a lattice point. Hence a function with the original divisor is F_Δ times a product of super theta functions as in Eq. (22). This completes the construction.

3. The Picard and Jacobian groups

In this section we compute explicitly the Picard, Jacobian, and divisor class groups of the super elliptic curve M. These objects were defined and discussed in [8], where they were all shown to be isomorphic in the normal case. Some but not all of the arguments used there apply more generally; nevertheless the isomorphisms will be verified here by direct calculation. We also exhibit the Abel map from M to its Jacobian, which is a projection rather than an isomorphism as for classical elliptic curves.

We consider the set of line bundles over the superelliptic curve M. A line bundle is specified by transition functions which are elements of $\mathcal{O}^*_{\text{ev}}$ [5,8,24], the invertible, even functions, on overlaps of charts. That is, the Picard group of line bundles under tensor product is $\text{Pic}(M) = H^1(M, \mathcal{O}^*_{\text{ev}})$ as usual. The standard exponential exact sheaf sequence,

$$0 \to \mathbb{Z} \to \mathcal{O}_{\text{ev}} \to \mathcal{O}^*_{\text{ev}} \to 1, \tag{24}$$

and the resulting cohomology sequence,

$$H^1(M, \mathbb{Z}) \to H^1(M, \mathcal{O}_{\text{ev}}) \to H^1(M, \mathcal{O}^*_{\text{ev}}) \to H^2(M, \mathbb{Z}), \tag{25}$$

imply as usual that the group of line bundles of degree zero is

$$\text{Pic}^0(M) = H^1(M, \mathcal{O}_{\text{ev}})/H^1(M, \mathbb{Z}). \tag{26}$$

We can also describe a line bundle by the set of divisors of all its meromorphic sections. Since the ratio of two sections is a function, this gives an isomorphism between $\text{Pic}^0(M)$ and $\text{Cl}^0(M)$, the group of classes of degree-zero divisors modulo divisors of meromorphic functions [5,8]. We will compute both groups explicitly, verifying this isomorphism and obtaining the projection map $\pi : M \to \text{Pic}^0(M)$.

The divisor class group can be computed immediately from the results of the previous section. We first observe that every divisor Δ of degree zero is equivalent to one of the form $P - P_0$ with P_0 a fixed basepoint on M, for example $(0, 0)$. This is because P can always be chosen so that $\Delta - P + P_0$ satisfies the sum rules (14) and is therefore the divisor of a function. What changes from the classical elliptic curve results is that the choice of P is not unique: evidently we are free to add multiples of δ to the even coordinate of P without changing the equivalence class of the divisor $P - P_0$. This establishes the central result of this section: the Abel map $\pi : M \to \text{Cl}^0(M)$ which sends a point P to the divisor class $[P - P_0]$ is a projection onto $\text{Cl}^0(M) \cong M/\equiv$, where the identification is $(z, \theta) \equiv (z + \alpha\delta, \theta)$. In the split case $\delta = 0$ we recover the naive isomorphism of M with $\text{Cl}^0(M)$ which might have been expected.

Before we confirm this result by direct computation of the Picard group, let us pause to explain in the context of the group structure why M cannot be isomorphic to its Picard group in general. The set of line bundles obviously carries the Abelian group structure given by tensor product. However, M carries no such group structure. Recall that M is the quotient of $\mathbb{C}^{1,1}$ by the discrete group G. Now, $\mathbb{C}^{1,1}$ itself can be identified with the nonabelian supertranslation group,

$$(z,\theta)\cdot(w,\chi)=(z+w+\theta\chi,\theta+\chi). \tag{27}$$

G is the discrete subgroup generated by $(1,0)$ and (τ,δ) acting by right multiplication. In view of the fact that [25]

$$(z,\theta)\cdot(\tau,\delta)\cdot(z,\theta)^{-1}=(\tau+2\theta\delta,\delta), \tag{28}$$

G is not a normal subgroup [15] and the quotient M does not inherit the group structure. However, $\mathbb{C}^{1,1}$ also admits an Abelian group structure via

$$(z,\theta)+(w,\chi)=(z+w,\theta+\chi). \tag{29}$$

Of course, M does not inherit this group structure either, because G is not a subgroup at all. But let us take the quotient $\mathbb{C}^{1,1}/\equiv$. On this quotient space G does act as a subgroup of the Abelian group structure, hence a normal subgroup, and the further quotient by G is the Picard group of M. [A subtlety arises here: $\equiv$ mods out by all $\alpha\delta$ with α in the Grassmann algebra Λ. This does not seem to include modding out by $\theta\delta$ as required to identify G as a subgroup. One must remember that the group laws are viewed as defined on the set of Λ-valued points to resolve the apparent paradox.] The moral is that the unexpected identification $\equiv$ really provides the minimal modification of M which will admit an Abelian group structure as $\mathrm{Pic}^0(M)$ must.

We now turn to the direct computation of $\mathrm{Pic}^0(M)$ from (26). It seems cleanest to compute $H^1(M,\mathcal{O}_{\mathrm{ev}})$ as the group cohomology $H^1(G,\mathcal{O}_{\mathrm{ev}})$ with values in the functions on $\mathbb{C}^{1,1}$, following similar calculations of Hodgkin [6,26]. For an explanation of the equivalence between the sheaf cohomology of M and the group cohomology of G, see [27]; the techniques of group cohomology we use are fairly intuitive and can be found in [2, Appendix B]. In particular, there is the exact sequence

$$0\to H^1[(S),\mathcal{O}_{\mathrm{ev}}^T]\to H^1(G,\mathcal{O}_{\mathrm{ev}})\to H^1[(T),\mathcal{O}_{\mathrm{ev}}], \tag{30}$$

where $(T),(S)$ are the cyclic subgroups generated by the two generators of G, and $\mathcal{O}_{\mathrm{ev}}^T$ are the T-invariant functions. The last cohomology group in this sequence is trivial, so we get the isomorphism

$$H^1(G,\mathcal{O}_{\mathrm{ev}})\cong H^1[(S),\mathcal{O}_{\mathrm{ev}}^T], \tag{31}$$

which we use for our computation. In geometric language this says that a torus is made from the plane by first making the cylinder with fundamental group (T), whose sheaf cohomology is trivial because it is noncompact. The cohomology of the torus is then computed directly from functions $\mathcal{O}_{\mathrm{ev}}^T$ on the cylinder by identifying its ends with S.

A cocycle for $H^1[(S),\mathcal{O}_{\mathrm{ev}}^T]$ is determined by assigning to the generator S a T-invariant function $F=f(z)+\theta\phi(z)$; it is trivial (exact) if $F=\tilde{F}-S\tilde{F}$ for some T-invariant function $\tilde{F}=g(z)+\theta\gamma(z)$. This requires

$$f(z)+\theta\phi(z)=g(z)+\theta\gamma(z)-g(z+\tau+\theta\delta)-(\theta+\delta)\gamma(z+\tau+\theta\delta), \tag{32}$$

which amounts to

$$
\begin{aligned}
f(z) &= g(z) - g(z+\tau) - \delta\gamma(z+\tau), \\
\phi(z) &= \gamma(z) - \gamma(z+\tau) - \delta g'(z+\tau).
\end{aligned} \tag{33}
$$

Because every function appearing here is T-invariant, which is to say periodic, they have Fourier series expansions of the form,

$$
f(z) = \sum_{n=-\infty}^{\infty} f_n e^{2\pi i n z}, \tag{34}
$$

and similarly for the other functions. Then the triviality of the cocycle becomes the conditions on the Fourier coefficients,

$$
\begin{aligned}
f_n &= g_n(1 - e^{2\pi i n\tau}) - \delta\gamma_n e^{2\pi i n\tau}, \\
\phi_n &= \gamma_n(1 - e^{2\pi i n\tau}) - 2\pi i n\delta g_n e^{2\pi i n\tau}.
\end{aligned} \tag{35}
$$

Given f_n and ϕ_n, these equations can always be solved for g_n and γ_n, *except* in the case $n = 0$ when the conditions for triviality are

$$
f_0 = -\delta\gamma_0, \quad \phi_0 = 0. \tag{36}
$$

That is, the nontrivial cocycles are precisely the odd constants and the even constants modulo multiples of δ: $H^1(M, \mathcal{O}_{\rm ev}) = \mathbb{C}^{1,1}/\equiv$, in agreement with Proposition 3 of [6].

To complete the calculation, we must compute $H^1(G,\mathbb{Z})$. Of course this is a lattice $\mathbb{Z}\oplus\mathbb{Z}$, but we need to know where this lattice sits inside $H^1(G,\mathcal{O}_{\rm ev})$. An element of $H^1(G,\mathbb{Z})$ assigns integers $-n, m$ to the generators T, S respectively. In the calculation above, however, we used the triviality of $H^1[(T),\mathcal{O}_{\rm ev}]$ to represent each class in $H^1(G,\mathcal{O}_{\rm ev})$ by a cocycle which assigned zero to the generator T. To find such a representative of our element of $H^1(G,\mathbb{Z})$, we pick a function $g(z)$ such that $-n = g(z) - g(z+1)$, for example $g(z) = nz$, and subtract the trivial cocycle which assigns

$$
\begin{aligned}
T &\mapsto g(z) - g(z+1) = -n, \\
S &\mapsto g(z) - g(z+\tau+\theta\delta) = -n\tau - n\theta\delta,
\end{aligned} \tag{37}
$$

obtaining the new representative

$$
T \mapsto 0, \quad S \mapsto m + n\tau + n\theta\delta. \tag{38}
$$

In terms of our identification $H^1(M,\mathcal{O}_{\rm ev}) = \mathbb{C}^{1,1}/\equiv$, the elements of $H^1(M,\mathbb{Z})$ are thus precisely the lattice points $m(1,0) + n(\tau,\delta)$ in $\mathbb{C}^{1,1}$. This explicitly shows that

$$
\mathrm{Pic}^0(M) = H^1(M,\mathcal{O}_{\rm ev})/H^1(M,\mathbb{Z}) = M/\equiv\ = \mathrm{Cl}^0(M). \tag{39}
$$

Next we wish to similarly calculate the Jacobian of M, defined [8] as the set of odd (Λ-)linear functionals on the holomorphic differentials of weight 1/2, modulo those functionals which are the periods of the differentials around cycles. A 1/2-differential on a super Riemann surface is a section of the canonical bundle, the bundle whose transition functions are the Berezinian determinants of those of M. Since supertranslations (1) have unit determinant, this bundle is trivial for superelliptic curves, and the 1/2-differentials can be identified with functions. The periods of such a function are obtained by integrating it over all homology cycles. Equivalently, we can lift a function F to the covering space $\mathbb{C}^{1,1}$ and find an antiderivative Φ with $F = D\Phi$; the periods are the changes in Φ under the covering transformations generated by T and S. The Jacobian is then the set of odd linear functionals on $H^0(M, \mathcal{O}) = \{a + \theta\alpha : \alpha\delta = 0\}$ modulo periods. Note that we consider all global functions, not merely even ones, so as to obtain a Λ-module rather than a Λ_{ev}-module.

The periods of the function $a + \theta\alpha$ are easily found. An antiderivative is $\Phi = \alpha z + \theta a$. Under the translation T this changes by α, while under the other generator S it changes by $\alpha\tau + \delta a$. The odd linear functionals which send $a + \theta\alpha$ to integral linear combinations of these two constants will be equivalent to zero in the Jacobian.

To understand the structure of the linear functionals on the functions $a + \theta\alpha$ let us begin with the simpler case in which δ is one of the generators of the Grassmann algebra Λ. Then the set of α which annihilate δ is just the set of multiples of δ, and a function $a + \theta\alpha$ is a linear combination of the functions 1 and $\theta\delta$. Then an odd linear functional is determined by sending 1 to some odd constant η, and sending $\theta\delta$ to some odd constant κ. By linearity, $\delta\kappa = 0$, so $\kappa = \delta k$ for an even constant k defined modulo δ. Hence we have found that the odd linear functionals correspond precisely to points (k, η) in $\mathbb{C}^{1,1}/\equiv$. They can be viewed as mapping $1 \mapsto \eta$ and $\theta \mapsto k$, just as if 1 and θ formed a basis for the functions, except that k is only defined modulo δ. Since the periods are just the familiar lattice points generated by $(1, 0)$ and (τ, δ), we have explicit agreement between the Jacobian and the Picard group computed earlier. One can easily verify that the isomorphism between them is the one described in [8]: given a line bundle in Pic^0, represent it by a divisor in the form $P - P_0 = (k, \eta) - (0, 0)$ and associate to it the linear functional which integrates a function from P_0 to P, which will also be (k, η) with our conventions.

What changes in the general case in which δ is not a generator of Λ? A linear functional is still determined by its effect on the functions of the forms a and $\theta\alpha$ separately. A functional on $\{a\}$ is still determined by the odd constant η which is the image of 1, but the functionals on $\{\theta\alpha\}$ are not so clear. We are asking for the Λ-linear functionals on the ideal $I = \{\alpha : \alpha\delta = 0\}$, the annihilator of δ. Because Λ is an example of a quasi-Frobenius, or self-injective ring [28], any such functional is multiplication by an even constant k [29] which is determined up to constants annihilating I. Again because Λ is self-injective, these are the multiples of δ [30]. Hence the isomorphism of the Picard and Jacobian groups holds in general. To see that Λ is indeed self-injective one can apply a simple test from [30]: the annihilator of the annihilator of any minimal ideal of Λ must be the ideal itself. The unique minimal ideal in the Grassmann algebra with generators $\beta_1, \beta_2, \ldots, \beta_N$ is the set of multiples of $\beta_1\beta_2\cdots\beta_N$; its annihilator is the ideal of all nilpotents, whose annihilator is indeed the minimal ideal again.

4. The group law in a projective embedding

As shown in [4], the superelliptic curve M can be embedded in the projective superspace $P^{3,2}$ with the help of the super Weierstrass function $R(z,\theta)$. Indeed, the map

$$(z,\theta) \mapsto (R, R', R'', 1; DR, D^3R) = (x, y, u, v; \phi, \psi) \tag{40}$$

in the affine chart $v = 1$, with the extension to the points at infinity,

$$(0,\theta) \mapsto (0,0,1,0;0,\theta), \tag{41}$$

embeds M as the locus of points satisfying the following homogeneous polynomial equations:

$$\begin{aligned}
&y^2v - 4x^3 + g_2xv^2 + g_3v^3 - 2\phi\psi v = 0,\\
&2y\psi v + (g_2v^2 - 12x^2)\phi + \delta\dot{g}_2xv^2 + \delta\dot{g}_3v^3 = 0,\\
&2yuv + (g_2v^2 - 12x^2)y - \delta\dot{g}_2\phi v^2 = 0,\\
&2(g_2v^2 - 12x^2)uv + (g_2v^2 - 12x^2)^2 + 2\delta\dot{g}_2\psi v^3 = 0,
\end{aligned} \tag{42}$$

where $g_2(\tau)$ and $g_3(\tau)$ are the usual modular functions. The last equation is redundant except when $y = 0$; M is not a complete intersection.

Now although M is a variety, it does not carry a group structure; its Jacobian, which does carry a group structure, is not a variety. [Sketch of proof: the functions on the Jacobian are the functions on M which are invariant under the identification $z \equiv z + \alpha\delta$. This does not include the super Weierstrass function R or its derivatives, but does include δR, δDR, $\delta R'$, ... However, none of these functions can be expressed as a polynomial (which would necessarily be a linear combination because $\delta^2 = 0$) in the others, so this "coordinate ring" is not finitely generated.] What then becomes of the standard geometric implementation of the group law by intersecting an elliptic curve with lines?

We attempt to follow the usual construction by taking a meromorphic function F on M given by

$$F = aR + R' + \alpha DR + \beta D^3R + b. \tag{43}$$

This is the restriction to M of a linear function on $P^{3,2}$ (in the chart $v = 1$),

$$F = ax + y + \alpha\phi + \beta\psi + bv. \tag{44}$$

The conditions for F to have a principal zero at some point on M, $F = DF = 0$, translate into the linear equations of a plane,

$$\begin{aligned}
&ax + y + \alpha\phi + \beta\psi + bv = 0,\\
&a\phi + \psi - \alpha y - \beta u = 0,
\end{aligned} \tag{45}$$

to be solved simultaneously with the equations of M. Note that this is hardly a generic plane, but rather a very special one encoding the notion of a principal zero. It is given by simple linear equations only because the embedding of M was constructed using the covariant derivative D which also encodes the superconformal structure. We can adjust the four parameters a, b, α, β so that F has principal simple zeros at any two given points $P_i = (z_i, \theta_i), i = 1,2$ on M. The naive expectation would be that F has a principal triple pole at $(0,0)$ and, as a consequence of our function theory, there is a third point of intersection with M at P_3 such that $P_1 + P_2 + P_3 = 0 \bmod \equiv$. This turns out to be wrong on two counts. First, using the fact that the singular part of $R(z,\theta)$ is $1/z^2$, we find for the singular part of F

$$\begin{aligned} F &\sim az^{-2} - 2z^{-3} - 2\alpha\theta z^{-3} + 6\beta\theta z^{-4} \\ &= (az - 2 - 3a\theta\beta + 2\theta\alpha)(z - \theta\beta)^{-3}, \end{aligned} \tag{46}$$

so that the triple pole is actually located at $(0,\beta)$. This is a consequence of the fact that the most singular term in F is the nilpotent $\beta D^3 R$ term. We could not have avoided this by including an equally singular even term R'' in F, since then the condition $DF = 0$ for a principal zero would involve $D^5 R$, which is not one of the projective coordinates in our embedding. Next, there will indeed be a third point of intersection, another simple zero of F at P_3, but there is also a fourth intersection at the location of the triple pole itself: $(0,\beta)$ embeds in $P^{3,2}$ as $(0,0,1,0;0,\beta)$, which is easily seen to satisfy the homogeneous Eqs. (45). Thus the group law is realized in the form

$$P_1 + P_2 + P_3 - 3(0,\beta) = 0 \bmod \equiv . \tag{47}$$

This is a translate of the standard group law, with the identity shifted to the point $(0, 3\beta)$ in the fiber of M at infinity. Note that the point which plays the role of the identity varies with the choice of points P_1, P_2 to be added, since β depends on this choice, but it can always be located geometrically as the fourth intersection of the curve with the plane. The existence of this fourth intersection could have been expected from the fact that the reduction of this embedding of M is not the usual degree 3 embedding of an elliptic curve in P^2, but the degree 4 embedding in P^3 using $\wp, \wp'$, and $\wp''$, in which there is indeed an extra intersection at infinity [31].

5. Isogenies

An isogeny of elliptic curves is a holomorphic map f from one to the other with the translation symmetry normalized out by requiring $f(0) = 0$. One proves that an isogeny is either constant or onto, and that it is always a homomorphism of the group structures. Since a super elliptic curve does not have a group structure, the super generalization will be that an isogeny $\mathbf{F}$ induces a group homomorphism via the projection maps to Pic^0:

$$\mathrm{Pic}^0(M_1) \xrightarrow{\pi_1^{-1}} M_1 \xrightarrow{\mathbf{F}} M_2 \xrightarrow{\pi_2} \mathrm{Pic}^0(M_2). \tag{48}$$

The homomorphism is independent of the inverse chosen for π_1. We will also discuss isogenies from a super elliptic curve to itself and show that only a split curve can admit nontrivial endomorphisms. This is due to a conflict between the linear nature of an isogeny and the quadratic constraint which is implicit in the superconformal structure of M.

Given two superelliptic curves $M_i = \mathbb{C}^{1,1}/G_i$ over Λ, with G_i generated by supertranslations of the form (1) with parameters τ_i, δ_i, an isogeny will be a holomorphic map $\mathbf{F} : M_1 \to M_2$ with $\mathbf{F}(0,0) = (0,0)$. (We will eventually require the map to be surjective as well.) Its lift to the covering space $\mathbb{C}^{1,1}$ takes the form,

$$(z,\theta) \mapsto \mathbf{F}(z,\theta) = [F(z,\theta), \Psi(z,\theta)] = [f(z) + \theta\phi(z), \psi(z) + \theta g(z)], \quad (49)$$

with $f(0) = \psi(0) = 0$. Note that an isogeny is not assumed to be superconformal, but merely holomorphic, even though the groups G_i act superconformally.

In order that the map (49) descend to the quotient spaces M_i, it is necessary and sufficient that acting on (z,θ) with a generator of G_1 must change $\mathbf{F}(z,\theta)$ by the action of some element of G_2, which must be independent of z by continuity and the discreteness of the group. Therefore, we have

$$F(z+1,\theta) = F(z,\theta) + k + l\tau_2 + l\Psi(z,\theta)\delta_2, \quad (50)$$

$$\Psi(z+1,\theta) = \Psi(z,\theta) + l\delta_2, \quad (51)$$

$$F(z+\tau_1+\theta\delta_1, \theta+\delta_1) = F(z,\theta) + m + n\tau_2 + n\Psi(z,\theta)\delta_2, \quad (52)$$

$$\Psi(z+\tau_1+\theta\delta_1, \theta+\delta_1) = \Psi(z,\theta) + n\delta_2, \quad (53)$$

with integers k, l, m, n. If we use (49) to write these conditions in terms of f, ϕ, ψ, g, we obtain

$$f(z+1) - f(z) = k + l\tau_2 + l\psi(z)\delta_2, \quad (54)$$

$$\phi(z+1) - \phi(z) = lg(z)\delta_2, \quad (55)$$

$$\psi(z+1) - \psi(z) = l\delta_2, \quad (56)$$

$$g(z+1) - g(z) = 0, \quad (57)$$

$$f(z+\tau_1) - f(z) = m + n\tau_2 + n\psi(z)\delta_2 - \delta_1\phi(z+\tau_1), \quad (58)$$

$$\phi(z+\tau_1) - \phi(z) = ng(z)\delta_2 - \delta_1 f'(z+\tau_1), \quad (59)$$

$$\psi(z+\tau_1) - \psi(z) = n\delta_2 - \delta_1 g(z+\tau_1), \quad (60)$$

$$g(z+\tau_1) - g(z) = -\delta_1\psi'(z+\tau_1). \quad (61)$$

The analysis of these equations is somewhat tedious, but straightforward. Eqs. (57) and (61) imply that $\delta_1 g(z)$ is an elliptic function, and entire, hence a constant. (A simple argument using the filtration of Λ shows that this is true even though τ_1 may have a nilpotent part.) Given this, Eqs. (56) and (60) say that $\psi'(z)$ is elliptic, hence constant. Calling the constant γ and using the normalization $\psi(0) = 0$, we have $\psi(z) = \gamma z$. According to (56), $\gamma = l\delta_2$. From (60),

$$\delta_1 g(z) = n\delta_2 - \gamma\tau_1 = (n - l\tau_1)\delta_2, \tag{62}$$

so that δ_2 must be a multiple of δ_1 [and vice versa if we assume $g(z)$ is invertible]. Consequently, multiplying any equation by δ_1 will kill terms containing either δ_i, and terms involving $\psi(z)\delta_i$ are already zero.

With this information, Eqs. (55) and (59) say that $\delta_1\phi(z)$ is elliptic, so constant. Then (54) and (58) say that $f'(z)$ is elliptic, which together with the normalization $f(0) = 0$ gives $f(z) = az$ where the constant $a = k + l\tau_2$. Eqs. (57) and (61) give that $g(z)$ is elliptic; so $g(z) = c$, a constant, and (55) and (59) make $\phi'(z)$ a constant, so $\phi(z) = \alpha z + \beta$ with $\alpha = lc\delta_2$ according to (55). Having expressed all the unknown functions in terms of a few constants, all eight equations are satisfied provided the constants satisfy a few relations. Eq. (58) requires $\delta_1\beta = m + n\tau_2 - a\tau_1$; Eq. (59) gives $\delta_1 a = nc\delta_2 - \alpha\tau_1 = (n - l\tau_1)c\delta_2$; and Eq. (60) implies $\delta_1 c = n\delta_2 - \gamma\tau_1 = (n - l\tau_1)\delta_2$. Collecting all these results, the general form of an isogeny is given by

$$\begin{aligned} &f(z) = az, \quad \phi(z) = \alpha z + \beta, \quad \psi(z) = \gamma z, \quad g(z) = c; \\ &(z,\theta) \mapsto [az + \theta(\alpha z + \beta),\ \gamma z + \theta c], \end{aligned} \tag{63}$$

where

$$\begin{aligned} &a = k + l\tau_2, \quad \gamma = l\delta_2, \quad \alpha = c\gamma, \\ &\delta_1 a = (n - l\tau_1)c\delta_2, \quad \delta_1 c = (n - l\tau_1)\delta_2, \quad \delta_1\beta = m + n\tau_2 - (k + l\tau_2)\tau_1. \end{aligned} \tag{64}$$

Having obtained this general form, we can use it to answer several questions about isogenies of super elliptic curves. First let us ask whether an isogeny, which is only holomorphic by definition, is in fact a superconformal map. A map $\mathbf{F}(z,\theta) = [F(z,\theta), \Psi(z,\theta)]$ is superconformal provided that $DF = \Psi D\Psi$; in our case this says that

$$\alpha z + \beta + \theta a = \gamma c z + \theta c^2. \tag{65}$$

This requires $\alpha = \gamma c$, which is one of the conditions (64); $a = c^2$, which need only hold modulo the annihilator of δ_1 according to (64); and $\beta = 0$, which is a completely new restriction. We conclude that not every isogeny is superconformal; the superconformal ones take the special form,

$$(z,\theta) \mapsto (c^2 z + \theta\gamma c z, \gamma z + \theta c). \tag{66}$$

Next, we see that while isogenies of ordinary elliptic curves are either constant or onto, this is not true for super elliptic curves. If the parameter a is nilpotent, for example, a nonconstant isogeny may have a constant reduction, so that it is not surjective. This is simply because the presence of nilpotents can lead to a wider range of singularities for maps in general. We prefer not to consider such singularities, so we assume from now on that all our isogenies are surjective, which requires that the reduced parameters a_{rd} and c_{rd} be nonzero. The important consequence of this is that δ_1 is a multiple of δ_2 as well as vice versa.

We now prove that a surjective isogeny of super elliptic curves induces a well-defined homomorphism of their Picard groups via the diagram (48),

$$\mathrm{Pic}^0(M_1) \xrightarrow{\pi_1^{-1}} M_1 \xrightarrow{\mathbf{F}} M_2 \xrightarrow{\pi_2} \mathrm{Pic}^0(M_2). \tag{67}$$

A point (z,θ) of $\mathrm{Pic}^0(M_1)$ is the image under π_1 of any point $(z+\epsilon\delta_1,\theta)$ of M_1 for any ϵ. The isogeny $\mathbf{F}$ sends this point to

$$(z+\epsilon\delta_1,\theta) \mapsto [az + a\epsilon\delta_1 + \theta(\alpha z+\beta) + \theta\alpha\epsilon\delta_1, \gamma z + \gamma\epsilon\delta_1 + \theta c] \tag{68}$$

in M_2. Then π_2 removes any multiple of δ_2 from the first coordinate. The result is indeed independent of ϵ, showing that the composite map is well-defined, because the surjectivity makes δ_1 a multiple of δ_2. This also eliminates the term $\gamma\epsilon\delta_1$ from the second coordinate, because the conditions (64) include $\gamma = l\delta_2$.

Now, at the level of the Picard groups, we can drop α, which is a multiple of δ_2, from (63) and write an isogeny as

$$(z,\theta) \mapsto (az+\theta\beta, \gamma z + \theta c). \tag{69}$$

But this is a linear map, and the group law is simply addition in these coordinates, so the map is a group homomorphism as claimed.

Next we examine isogenies of a super elliptic curve M onto itself (endomorphisms). Setting $\tau_1 = \tau_2 = \tau$, $\delta_1 = \delta_2 = \delta$ in the general formulas, we obtain in this case

$$(z,\theta) \mapsto [az + \theta(\alpha z+\beta), \gamma z + \theta c], \tag{70}$$

with

$$\delta c = \delta(n - l\tau), \qquad \delta a = \delta c^2, \tag{71}$$

$$a = k + l\tau, \quad \gamma = l\delta, \quad \alpha = cl\delta, \quad \delta\beta = m + n\tau - (k+l\tau)\tau. \tag{72}$$

In the special case when M is split, $\delta = 0$, we lose the conditions (71) and obtain the simple form,

$$(z,\theta) \mapsto (az+\theta\beta, \theta c), \tag{73}$$

$$a = k + l\tau, \quad 0 = m + n\tau - (k+l\tau)\tau. \tag{74}$$

In particular, c is now arbitrary; there is no relation like $a = c^2$ in this case.

We see that in the split case, multiplication by an integer k, $(z,\theta) \mapsto (kz, k\theta)$, is an endomorphism, which was to be expected since M and its Picard group coincide in this case. But this is not true more generally, since this map violates the condition $\delta a = \delta c^2$, which is a vestige of the superconformal action of the group G. In fact, for $\delta \neq 0$, this implies $a_{\mathrm{rd}} = c_{\mathrm{rd}}^2$, which gives the quadratic constraint,

$$l^2\tau_{\mathrm{rd}}^2 - (2n+1)l\tau_{\mathrm{rd}} + (n^2 - k) = 0. \tag{75}$$

This must hold in addition to the usual quadratic constraint appearing in the theory of complex multiplication, which here arises from reducing the condition on $\delta\beta$ in (72),

$$l\tau_{\rm rd}^2 + (k-n)\tau_{\rm rd} - m = 0. \tag{76}$$

When $l \neq 0$ we are indeed describing complex multiplication, meaning an endomorphism with a complex. By eliminating the quadratic term between these equations, we conclude that $\tau_{\rm rd}$ is rational, not complex, a contradiction which shows that a nonsplit M cannot admit complex multiplication. However, even in the case $l = 0$ when a is an integer, the constraints give $k = n^2$ in addition to the usual $k = n$ and $m = 0$, so that M admits only the trivial endomorphisms $k = n = 0, 1$.

6. Supercommuting differential operators from super elliptic curves

The beautiful Krichever theory which produces solutions to the Kadomtsev–Petviashvili (KP) hierarchy of nonlinear PDEs from geometric data consisting of a line bundle over an algebraic curve together with some coordinate choices is by now well-known [32,33]. The simplest explicit example uses a line bundle $\mathcal{L}$ of degree zero over an elliptic curve M to construct the commuting pair of ordinary differential operators,

$$Q = \frac{d^2}{dx^2} - 2\wp(x+a), \tag{77}$$

$$P = Q_+^{3/2} = \frac{d^3}{dx^3} - 3\wp(x+a)\frac{d}{dx} - \frac{3}{2}\wp'(x+a), \tag{78}$$

where $Q_+^{3/2}$ is the differential operator part of $Q^{3/2}$ computed in the larger algebra of formal pseudodifferential operators. The correspondence which associates Q and P to the meromorphic functions $\wp(z)$ and $-\wp'(z)/2$ on M respectively sets up an isomorphism between the commutative ring of differential operators generated by Q, P and the ring of meromorphic functions on M with poles only at $z = 0$, which is generated by $\wp(z)$ and $-\wp'(z)/2$. As $\mathcal{L}$ varies through the Picard group $\mathrm{Pic}^0(M)$, the parameter a changes and the ring of operators is isospectrally deformed. In fact, there is an infinite set of linear coordinates t_n for $\mathrm{Pic}^0(M)$ on which a depends linearly, with Q satisfying the KP equations,

$$\partial Q/\partial t_n = [Q_+^{n/2}, Q]. \tag{79}$$

The corresponding construction of solutions to the supersymmetric KP hierarchies was worked out recently [10,11,14]. One surprise was that the geometric data involve a line bundle over a specific type of algebraic supercurve, which cannot be a super Riemann surface except in the special case of genus one. Another was the fact that linear flow in the Picard group of a fixed supercurve is described by a new super KP hierarchy discovered by Mulase and myself, and not by either of the previously known hierarchies due to Manin–Radul or to Kac–van de Leur. It follows that explicit solutions to this new super KP hierarchy can be constructed using the information about the Picard group of a super elliptic curve developed in the previous sections. In this section we exhibit and discuss these solutions. We change our notation slightly to conform to the conventions of the literature on KP theory: the standard coordinates on the covering space $\mathbb{C}^{1,1}$ of the supertorus M will now be denoted by (w, ϕ), so that (z, θ) can be reserved for a different set of local coordinates on M to be introduced below.

We begin with an overview of the construction to be carried out. In a small disk U around the point $P_0 : (w, \phi) = (0,0)$ we introduce new coordinates (z, θ) such that z^{-2} and θz^{-3} (other exponents would work as well) extend to global holomorphic functions on $M - P_0$. We fix a nontrivial line bundle $\mathcal{L}$ of degree zero on M and note that it is holomorphically trivial on each of the Stein patches U and $M - U$, hence completely described by a transition function across the overlap, a small annular neighborhood of ∂U, which we can take to be the circle $|z| = 1$. We embed $\mathcal{L}$ in a family of bundles $\mathcal{L}(x, \xi)$ by multiplying its transition function by an extra factor $\exp(xz^{-1} + \xi\theta)$. Although these bundles have no holomorphic sections, they have one which has the form $(z^{-1}+$ holomorphic$)$ near P_0 (note that this is different from having a principal simple pole there); the expression of this section in the coordinates (z, θ) in the chart $M - U$ is the Baker-Akhiezer function $B(z, \theta, x, \xi)$ [although we will express it in terms of the covering space coordinates (w, ϕ) instead]. It is the basic object in the theory and we will construct it explicitly in terms of super theta functions. We observe that successive derivatives of B with respect to x and ξ produce sections having poles of higher orders at P_0 and constitute a basis for the space of meromorphic sections on M with poles only at P_0. This allows us to set up an isomorphism between the ring of functions having poles only at P_0 and a ring of super differential operators as follows. Given such a meromorphic function F, FB is a section with poles at P_0 only, so it must be a linear combination of derivatives of B. But this is to say that it arises from B by the action of a certain differential operator O_F, so we associate this operator to F. It can be computed for an explicit F by matching the singular and constant terms in the Laurent expansions of FB and $O_F B$ about P_0. We will exhibit a set of generators for this ring analogous to Q, P above, and discuss how they flow under the deformations of $\mathcal{L}$ described by the super KP equations.

We start with the specification of the new coordinates (z, θ). In order that z^{-2} extend to a holomorphic function away from P_0, we choose

$$z^{-2} = R(w, \phi) + \tfrac{2}{3}c = \wp(w; \tau + \phi\delta) + \tfrac{2}{3}c, \tag{80}$$

where c is a constant and R is the super Weierstrass function introduced earlier. Similarly, in order that θz^{-3} extend holomorphically we use a function with behavior ϕw^{-3} near P_0, setting

$$-2\theta z^{-3} = DR(w, \phi) - 2\gamma = \delta\dot{\wp}(w; \tau) + \phi\wp'(w; \tau) - 2\gamma, \tag{81}$$

where γ is another constant, and we recall that a dot means ∂_τ while a prime denotes ∂_w. Using the Laurent expansion of $\wp(w)$ [18] we obtain the relation between the two sets of coordinates,

$$z^{-1} = w^{-1}[1 + \tfrac{1}{3}cw^2 + (\tfrac{1}{40}g_2 - \tfrac{1}{18}c^2 + \tfrac{1}{40}\dot{g}_2\phi\delta)w^4 + \cdots], \tag{82}$$

$$\theta = \phi - c\phi w^2 + \gamma w^3 + (\tfrac{1}{2}c^2 - \tfrac{1}{8}g_2)\phi w^4 - (c\gamma + \tfrac{1}{40}\dot{g}_2\delta)w^5 + \cdots. \tag{83}$$

Following the construction of the Baker function in the non-super theory [32], we express it as a ratio of super theta functions times a prefactor which is the exponential of a function with the behavior $xz^{-1} + \xi\theta = xw^{-1} + \xi\phi +$ holomorphic terms. Such a prefactor is

$$\exp[x\partial_w \log H(w,\phi) + \xi\phi]. \tag{84}$$

It has the correct singular part because of

$$\partial_w \log H(w,\phi) = w^{-1} + (q + \phi\delta\dot{q})w + \cdots, \tag{85}$$

where q is still the ratio of theta constants introduced in (9). It is invariant under the covering transformation $T: w \to w+1,\ \ \phi \to \phi$, while under the other generator S it acquires a phase

$$\exp(-2\pi i x + \xi\delta) = \exp -2\pi i(x - \xi\delta/2\pi i). \tag{86}$$

As our "pre-Baker function" $\hat{B}$ we take the product of this with a ratio of theta functions transforming by the opposite phase, namely

$$\hat{B} = \exp[x\partial_w \log H(w,\phi) + \xi\phi]\frac{H(w - a - \phi\alpha - x + \xi\delta/2\pi i, \phi + \alpha)}{H(w - a - \phi\alpha, \phi + \alpha)}, \tag{87}$$

as is easily verified using (21).

The parameters a and α describe the given line bundle $\mathcal{L}$: its divisor is $(a,\alpha) - (0,0)$. It has a section given by 1 outside the disk U, and $z^{-1}H(w - a - \phi\alpha, \phi + \alpha)$ inside. Equivalently, its transition function across ∂U (inside to outside) is $z/H(w-a-\phi\alpha, \phi+\alpha)$. Then the transition function of the deformed bundle $\mathcal{L}(x,\xi)$ is

$$\frac{z\exp(xz^{-1} + \xi\theta)}{H(w - a - \phi\alpha, \phi + \alpha)}. \tag{88}$$

Now $\hat{B}$ is to be viewed as a section of this bundle in the outside chart $M - U$; dividing by the transition function gives the same section in the inside chart U as a nonvanishing holomorphic function (the mismatch between the exponential factors) times $z^{-1}H(w-a-\phi\alpha-x+\xi\delta/2\pi i, \phi+\alpha)$, from which we see that the deformed bundle has divisor $(a + x - \xi\delta/2\pi i, \alpha) - (0,0)$. (We assume that all constants and parameters are small enough that the supports of these divisors are inside U.) In particular, x shifts the even coordinate of $\mathrm{Pic}^0(M)$ linearly and could be viewed as such a coordinate itself, but ξ does *not* shift the odd coordinate α. In fact, in view of the identification $\equiv$, ξ induces no flow on the Picard group at all but only changes the trivialization of the bundle.

The pre-Baker function can be normalized so that, apart from the exponential prefactor, its Taylor series in powers of w and ϕ begins with constant term unity. We will need this series through the quadratic terms in order to match singular parts of Laurent series later:

$$\begin{aligned}\hat{B}_n &= \frac{\Theta(a;\tau+\alpha\delta)}{\Theta(a+x-\xi\delta/2\pi i;\tau+\alpha\delta)}\hat{B} \\ &= \exp[x\partial_w \log H(w,\phi)+\xi\phi]\{1+\phi\alpha L'+\phi\delta\dot{L}-wL'-\phi\alpha w(L''+L'^2) \\ &\quad -\phi\delta w(\dot{L}'+L'\dot{L})+\tfrac{1}{2}w^2(L''+L'^2)+\tfrac{1}{2}\phi\alpha w^2(L'''+3L'L''+L'^3) \\ &\quad +\tfrac{1}{2}\phi\delta w^2(\dot{L}''+2L'\dot{L}'+L''\dot{L}+\dot{L}L'^2)+\cdots\}/N, \end{aligned} \tag{89}$$

where we have introduced the abbreviations

$$L=L(x,\xi,\tau,\delta)=\log\Theta(a+x-\xi\delta/2\pi i;\tau+\alpha\delta), \quad L'=\partial_x L, \quad \dot{L}=\partial_\tau L, \tag{90}$$

and the normalization constant N is the series in braces with x and ξ set to zero. Although the series has constant term unity, leading to the behavior $1/z$ for this section near P_0, we see that there are also terms proportional to ϕ, leading to additional singularities like ϕ/z. To obtain the true Baker function, we must subtract these off. Because of the exponential prefactor, derivatives of $\hat{B}_n$ with respect to x or ξ produce new sections[2] containing additional factors z^{-1} and ϕ respectively, so $\partial_\xi \hat{B}_n$ has a ϕ/z singularity. Subtracting the appropriate multiple of this yields the true Baker function,

$$\begin{aligned} B &= \hat{B}_n+(\alpha L'+\delta\dot{L})\partial_\xi\hat{B}_n \\ &= e^{[\cdots]}N^{-1}\Big\{1-wL'+\alpha\phi wL''+\delta\phi w\dot{L}'+\frac{\alpha\delta}{2\pi i}wL'L''+\tfrac{1}{2}w^2(L''+L'^2) \\ &\quad +\tfrac{1}{2}\phi\alpha w^2(L'''+2L'L'')+\tfrac{1}{2}\phi\delta w^2(\dot{L}''+2L'\dot{L}') \\ &\quad -\frac{\alpha\delta}{4\pi i}w^2(L'L'''+2L''L'^2)+\cdots\Big\}. \end{aligned} \tag{91}$$

It is now tedious but straightforward to work out the explicit correspondence between meromorphic functions F on M holomorphic away from P_0 and differential operators O_F in x,ξ by matching the singular terms in the series for $FB=O_F B$. For example, the operator corresponding to the super Weierstrass function $R(w,\phi)$, with a double pole at P_0, has the form $Q=d^2+\omega\partial+u$, with

$$\begin{aligned} \omega &= 2\left(\alpha\wp'(a+x;\tau)+\frac{\alpha\delta\xi}{2\pi i}\wp''+\delta\dot{\wp}\right), \\ u &= 2\left(-\wp+\frac{\xi\delta}{2\pi i}\wp'-\alpha\delta\dot{\wp}+\alpha\delta\dot{q}+\frac{\alpha\delta}{2\pi i}[\wp'\partial_x\log\Theta-(\wp-q)^2]\right), \end{aligned} \tag{92}$$

where all the functions have the same arguments as $\wp'(a+x;\tau)$, all odd parameters having been explicitly expanded out, and $d=\partial_x$, $\partial=\partial_\xi$. It follows from the general

[2] It may not be clear that derivatives of $\hat{B}$ are still sections of $\mathcal{L}(x,\xi)$. The point is that $\hat{B}$ is a global function on $M-U$ for all x,ξ, so its derivatives are too. They must extend into U as meromorphic sections since no essential singularity has been introduced.

theory, and can be verified explicitly, that the function $-R'(w,\phi)/2$ having a triple pole must correspond to $P = Q_+^{3/2}$. For any such second-order operator Q, one finds

$$P = Q_+^{3/2} = d^3 + \tfrac{3}{2}\omega\partial d + \tfrac{3}{2}ud + \tfrac{3}{4}\omega'\partial + \tfrac{3}{4}u'. \tag{93}$$

A set of generators for the ring of functions holomorphic off P_0 must contain an odd function in addition to $R, -R'/2$; this is conveniently taken to be $\sigma(w,\phi)$ of Eq. (10), which corresponds to the simple first-order operator

$$\Sigma = \partial + \frac{\delta}{2\pi i}d. \tag{94}$$

The supercommutativity of the generators Q, P, Σ of the isomorphic ring of operators can be verified explicitly. Although it is not manifest from the form of (92), both Q and P depend on x,ξ only through the combination $x - \xi\delta/2\pi i$ [see (86),(87) for the origin of this], and this is precisely the statement that they commute with Σ. We also have $\Sigma^2 = 0$. The vanishing of $[Q,P]$ leads to a pair of third-order differential equations for ω, u, namely

$$\omega_{xxx} + 3\omega\omega_{x\xi} + 6\omega u_x + 6u\omega_x + 3\omega_x\omega_\xi = 0, \tag{95}$$

$$u_{xxx} + 3\omega u_{x\xi} + 3\omega_x u_\xi + 6uu_x = 0. \tag{96}$$

One finds that, exactly as in the non-super case, the first equation is satisfied in virtue of the identity

$$\wp''' = 12\wp\wp' \tag{97}$$

satisfied by the Weierstrass function. However, the second equation requires, in addition to this identity, the relation

$$g_2 = 12(q^2 - 2\pi i\dot{q}) \tag{98}$$

between the modular function g_2 and the theta constant q. I have found similar relations in the literature on elliptic functions, though not in just this form; however, it is a simple consequence of the fact that the theta function satisfies the heat equation [18],

$$4\pi i\dot{\Theta}(w;\tau) = \Theta''(w;\tau). \tag{99}$$

As a consequence, its logarithm $f = \log\Theta$ satisfies

$$4\pi i\dot{f} = f'' + f'^2. \tag{100}$$

From the relation (9) between Θ and $\wp$ we get the Laurent expansion

$$f' = w^{-1} + qw - \tfrac{1}{60}g_2w^3 + \cdots, \tag{101}$$

and the desired relation (98) follows by using this in (100) and equating the coefficients of w^2 on both sides. This illustrates that the super KP system contains information about the modular dependence of the theta functions, through the coupling between τ and θ

in the superelliptic functions, which does not appear in the solutions to ordinary KP (although changes in moduli do figure in the additional symmetries of the KP hierarchy).

Finally, we describe the flows on the Picard group (further deformations of $\mathcal{L}$) which lead to the super KP equations for Q. These depend on an infinite set of parameters t_n which are (Grassmann) even or odd for even or odd n respectively. They multiply the transition function of $\mathcal{L}(x,\xi)$ by an additional factor $\exp t_{2n} z^{-n}$ or $\exp t_{2n+1}\theta z^{-n}$ respectively. At this point the properties of the new coordinates (z,θ) become important. Because z^{-2} extends to a holomorphic function on $M-U$, all the flows $t_4, t_8, \ldots$ are trivial since they can be undone by a change of bundle trivialization on $M-U$. Because θz^{-3} extends holomorphically, the same is true for $t_7, t_{11}, \ldots$. The parameter t_2 can be identified with x, since they produce the same deformation. The first nontrivial even flow is by $\exp t_6 z^{-3}$, and we need to understand the Baker function for the new bundle this produces. It should have an exponential prefactor having this singular behavior. For this purpose we employ the function $-R'(w,\phi)/2$ with singular part $w^{-3} = z^{-3} - cz^{-1} + \cdots$. Thus we need only multiply our previous Baker function by $\exp -t_6 R'/2$ and replace x by $x + ct_6$ to obtain the new one. The effect on the resulting differential operators is the replacement $a \to a + ct_6$ showing explicitly the flow on the Jacobian where a is the even coordinate. The flow would be trivial if we had chosen $c=0$; the motivation for introducing this constant is precisely to get a nontrivial t_6 flow.

Similarly, for the first nontrivial odd flow by $\exp t_3\theta z^{-1}$ an exponential prefactor with this behavior is

$$\exp t_3\partial_\eta \log H(w-\phi\eta, \phi+\eta) = \exp t_3 \left(\phi\frac{\Theta'(w;\tau)}{\Theta(w;\tau)} + \delta\frac{\dot{\Theta}(w;\tau)}{\Theta(w;\tau)}\right). \tag{102}$$

This function is invariant under the generator T, but acquires a phase

$$\exp -t_3(\pi i\delta + 2\pi i\phi) \tag{103}$$

under S. To obtain a well-defined pre-Baker function we compensate this phase by shifting the parameter α in the numerator factor

$$H(w - a - \phi\alpha - x + \xi\delta/2\pi i, \phi + \alpha) \tag{104}$$

in (87) by $\alpha \to \alpha - t_3$, which is the flow on the Jacobian in this case. The next odd flow $t_5\theta z^{-2}$ is actually trivial because there is a global function with this behavior, namely

$$-t_5 D\partial_w \log H(w,\phi) = t_5[\theta z^{-2} + \theta(\tfrac{1}{3}c - q) + \cdots]. \tag{105}$$

The mechanics of this triviality is rather interesting: if this function is used to form an exponential prefactor for the pre-Baker function, a shift of the parameter ξ will be required due to the term proportional to θ. We know that ξ only changes the trivialization of a bundle, and indeed the change in $\hat{B}$ resulting from this shift is subtracted off along with the ϕ/z poles in forming B, so that the differential operators are unchanged.

The higher flows can all be computed in the same manner. Because there are global functions with leading singularities z^{-n} and θz^{-n} for all $n \geq 2$, we can use them as prefactors for the Baker function (that is, to change the bundle trivialization in $M - U$) until any flow is reduced to a linear combination of those for $n = 1$. (In other words, any deformation can be reduced to a linear combination of the single even and odd generators for Pic^0.) Then its effect can be read off as a linear shift in the Jacobian coordinates a and α. It is not always true, however, that the even flows only shift a while the odd flows only shift α. In general each flow can shift both in the nonsplit situation. The flow parametrized by t_{10}, for example, acts by

$$a \to a + (\tfrac{1}{8}g_2 + \tfrac{5}{6}c^2)t_{10}, \quad \alpha \to \alpha - \tfrac{1}{8}\dot{g}_2\delta t_{10}, \tag{106}$$

showing how the supermodulus δ permits a flow in both even and odd coordinates.

The differential operator $Q + \frac{2}{3}c$ corresponding to the function z^{-2}, with its parameters shifted in this manner, gives a solution to the new super KP hierarchy of [10,11]. Unfortunately, unlike the standard KP theory, this hierarchy has no simple formulation in terms of Q itself, but is written in the Sato form in terms of the wave pseudodifferential operator K which satisfies

$$B(z,\theta,x,\xi,t) = K \exp\left(xz^{-1} + \xi\theta + \sum_{n=1}^{\infty}(t_{2n}z^{-n} + t_{2n+1}\theta z^{-n})\right), \tag{107}$$

conjugates Q into a simple form,

$$Kd^2K^{-1} = Q + \tfrac{2}{3}c, \tag{108}$$

and satisfies the super KP hierarchy

$$\partial K/\partial_{t_{2n}} = -(Kd^nK^{-1})_-K, \tag{109}$$

$$\partial K/\partial_{t_{2n+1}} = -(K\partial d^nK^{-1})_-K. \tag{110}$$

I have not tried to obtain an explicit expression for K.

7. Conclusions

In this paper we have developed the theory of super elliptic curves with an emphasis on the role of the supermodulus δ and the non-freely generated character of the cohomology modules. We discussed the building blocks for superelliptic functions, the super Weierstrass and theta functions, and proved the necessary and sufficient conditions for a divisor to be the divisor of a superelliptic function. We computed the Picard, Jacobian, and divisor class groups of a superelliptic curve, explicitly verifying the isomorphisms between them, and found that the Abel map $\pi : M \to \mathrm{Pic}^0(M)$ was a projection in the nonsplit case. The agreement between the different methods of calculation—cohomology for the Picard group, duality of modules for the Jacobian, function theory for the divisor class group—is very satisfying. We showed that the group law can be implemented in a projective embedding by intersecting M with planes chosen to encode the notion of principal zero, modulo the kernel of π and an ambiguity in the group identity element. We determined the general form of an isogeny of superelliptic curves, proving that it always induces a homomorphism of their Picard groups, and that a nonsplit curve admits trivial endomorphisms only. Finally, we applied this machinery to the explicit calculation of the supercommutative rings of differential operators which constitute the solution to the new super KP hierarchy corresponding to flow in the Jacobian of a superelliptic curve. The Baker function was expressed in terms of super theta functions and used to work out the differential operators corresponding to simple superelliptic functions, generalizing the classical Q, P pair of ordinary KP theory.

It would be natural to seek extensions of this theory in two directions: higher-genus super Riemann surfaces, and supercurves of genus one (and higher) which are not super Riemann surfaces. For super Riemann surfaces of higher genus the primary motivation is again to understand the consequences of the non-freely generated cohomology. One should again construct the Picard, Jacobian, and divisor class groups as explicitly as possible and check their isomorphism in the general nonsplit case. An Abel map from the surface to its Jacobian should be constructed and investigated. Function theory on the surface should be studied in terms of suitable super theta functions. A higher-genus analogue of the simple substitution $\tau \rightarrow \tau + \theta\delta$ which converts ordinary theta functions to super ones should be found. An important question is, On what space are the super theta functions naturally defined? The expectation that they should be defined on the Jacobian and then pulled back to M via the Abel map is not borne out by the genus-one situation studied here. Next, the duality properties of modules which determine the structure of the Jacobian should be understandable on the basis of Λ being self-injective, and this should be used to develop Serre duality for cohomology groups as Λ-modules rather than as $\mathbb{C}$-modules as in [34]. One should study the Krichever map from super Riemann surfaces with local coordinates to states in the operator formalism.

The motivation for studying genus-one supercurves which are not super Riemann surfaces, or Abelian supergroups on two generators whose action on $\mathbb{C}^{1,1}$ need not be superconformal, is to construct more general solutions to super KP hierarchies. (One should also find nontrivial endomorphisms of such curves with the relaxing of the superconformal constraint.) We know from [10] that the Manin–Radul and Kac–van de Leur super KP hierarchies describe simultaneous deformations of the supercurve M and the line bundle $\mathcal{L}$ over it, specifically by changing the patching of the coordinate θ along with that of the line bundle across ∂U. Even if M is initially a super Riemann surface, this property will not be preserved by the flow. Hence one needs to repeat enough of the analysis of this paper for general genus-one curves to construct the Baker functions for families of line bundles over such curves. One may learn something about where the locus of super Riemann surfaces sits inside the larger moduli space of genus-one curves by studying the corresponding super KP solutions, e.g., what is special about the rings of differential operators when M admits a superconformal structure? Without the covariant derivative D one will have to settle for Cartier divisors which are not sums of points. On the other hand one may be able to exploit the remarkable correspondence [35] between general supercurves and untwisted $N = 2$ super Riemann surfaces, and the resulting involution in the moduli space under which $N = 1$ super Riemann surfaces are fixed points. Perhaps this involution plays a role in the super KP theory.

Appendix A

A natural question is whether anything of number-theoretic interest results from seeking rational points on super elliptic curves. By analyzing a simple example we will see that this essentially amounts to finding rational points on the (co)tangent plane—more generally, the jets—of an ordinary elliptic curve at a rational point. This is disappointing, since rational points on planes are abundant and easy to find. However, as pointed out to me by J. Silverman, the question of counting such points having heights less than some bound is of interest.

When we consider super elliptic curves over $\mathbb{Q}$, the generators of the lattice cannot always be reduced to the form (1). Instead we must consider the more general form,

$$\begin{aligned} &T : z \to z + \omega_1 + \theta\delta_1, \quad \theta \to \theta + \delta_1, \\ &S : z \to z + \omega_2 + \theta\delta_2, \quad \theta \to \theta + \delta_2, \end{aligned} \tag{A.1}$$

with $\delta_1\delta_2 = 0$. As in [4], we find that the affine part of the super elliptic curve is embedded in $\mathbb{C}^{2,2}$ by the map,

$$(z, \theta) \mapsto (R, R'; DR, D^3R) = (x, y; \phi, \psi), \tag{A.2}$$

where $R(z,\theta) = \wp(z; \omega_1 + \theta\delta_1, \omega_2 + \theta\delta_2)$, as the set of solutions of the polynomial equations,

$$\begin{aligned} &y^2 - 4x^3 + g_2x + g_3 - 2\phi\psi = 0, \\ &2y\psi - (12x^2 - g_2)\phi + \sum_{i=1}^{2} \delta_i(\partial_{\omega_i}g_2x + \partial_{\omega_i}g_3) = 0. \end{aligned} \tag{A.3}$$

We fix Λ to be the Grassmann algebra on just two generators β_1, β_2, and consider the affine supertorus in $\mathbb{C}^{2,2}$ given by the equations,

$$\begin{aligned} &y^2 - 4x^3 + g_2x + g_3 - 2\phi\psi = 0, \\ &2y\psi - (12x^2 - g_2)\phi + a\beta_1x + b\beta_2 = 0, \end{aligned} \tag{A.4}$$

where g_2, g_3, a, b are rational. Now Λ is a four-dimensional vector space, and using the basis $\{1, \beta_1, \beta_2, \beta_1\beta_2\}$ we can write

$$\begin{aligned} &x = x_{\mathrm{rd}} + x_{12}\beta_1\beta_2, \qquad y = y_{\mathrm{rd}} + y_{12}\beta_1\beta_2, \\ &\phi = \phi_1\beta_1 + \phi_2\beta_2, \qquad \psi = \psi_1\beta_1 + \psi_2\beta_2. \end{aligned} \tag{A.5}$$

By a rational point we understand one whose components in this basis are rational. Inserting these expressions into the polynomial Eqs. (A.4), we obtain

$$y_{\mathrm{rd}}^2 - 4x_{\mathrm{rd}}^2 + g_2 x_{\mathrm{rd}} + g_3 = 0, \tag{A.6}$$

$$2y_{\mathrm{rd}}\psi_1 - (12x_{\mathrm{rd}}^2 - g_2)\phi_1 = -ax_{\mathrm{rd}} - b, \tag{A.7}$$

$$2y_{\mathrm{rd}}\psi_2 - (12x_{\mathrm{rd}}^2 - g_2)\phi_2 = 0, \tag{A.8}$$

$$2y_{\mathrm{rd}}y_{12} - (12x_{\mathrm{rd}}^2 - g_2)x_{12} = 2(\phi_1\psi_2 - \phi_2\psi_1). \tag{A.9}$$

The first equation says that $(x_{\mathrm{rd}}, y_{\mathrm{rd}})$ must be a rational point on the reduced elliptic curve. The three remaining equations share a common structure which can be understood by recalling the invariant differential of the reduced curve,

$$2y_{\mathrm{rd}}dy_{\mathrm{rd}} - (12x_{\mathrm{rd}}^2 - g_2)dx_{\mathrm{rd}} = 0. \tag{A.10}$$

This can be viewed as defining a linear map from rational values of dx_{rd} to rational values of dy_{rd}, or vice versa, at the chosen rational point of M_{rd}; the derivative map defined over the rationals. Similarly here we get a map from rational (ϕ_1, ϕ_2, x_{12}), playing the role of dx_{rd}, to rational (ψ_1, ψ_2, y_{12}) analogous to dy_{rd}, which is a deformation of the derivative map and is computed by solving linear equations only. A Grassmann algebra with more generators will bring higher derivatives into play through later terms in the Taylor expansions of the Eqs. (A.4).

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